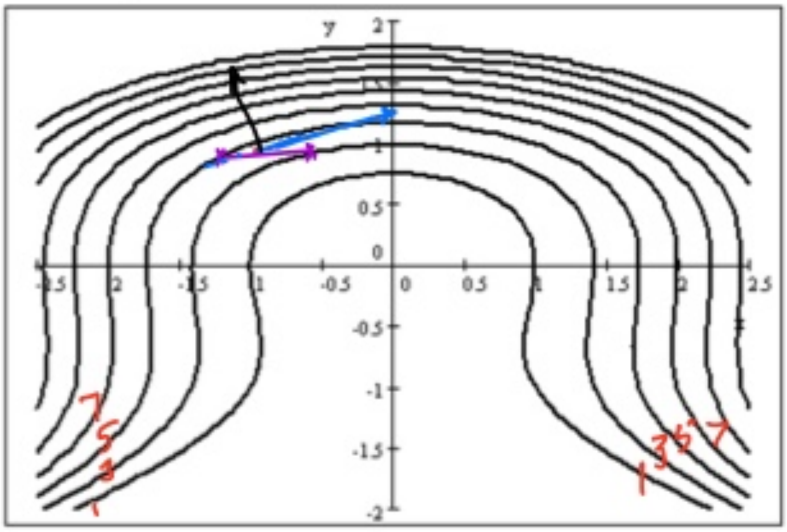
2.25 In the contour diagram below of the function $G(x, y)$ the contour nearest the origin has the value 1, and the values of $G(x, y)$ at the contours increase by 2 as y increases from the origin.



(a) Estimate $G(-1, 1)$. ≈ 4

(b) Estimate $\frac{\partial G}{\partial x}(-1, 1)$. $\approx \frac{\Delta G}{\Delta x} \approx \frac{3-5}{.7} \approx \frac{-2}{.7} \approx 3$

(c) Find a unit vector u so that $\frac{\partial G}{\partial u}(-1, 1) = 0$. $u \approx \frac{(1, .25)}{\sqrt{1+(.25)^2}} \approx (.9, .2)$ (also $(-.9, -.2)$)

(d) Find a unit vector v so that the instantaneous rate of change of G in direction v is the greatest possible at $(x, y) = (-1, 1)$.

direction of ∇G $\quad u = \frac{\nabla G}{|\nabla G|} \approx (-.2, .9)$

Warmups ① Calculate $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix}$

② Find the derivative matrix of

$$F(x,y,z) = \left(x\ln(xyz),\ \arccos(x3^y)\right)$$

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①

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$2\times 3 \quad 3 \times 4$

$$= \begin{pmatrix} 4 & 3 & 1 & 2 \\ 2 & 3 & 3 & 0 \end{pmatrix}$$

② $F(x,y,z) = (\underbrace{x\ln(xyz)}_{F_1}, \underbrace{\arccos(x3^y)}_{F_2})$

$$F' = \begin{pmatrix} - \nabla F_1 - \\ - \nabla F_2 - \end{pmatrix} = \begin{pmatrix} \left(\ln(xyz) + x\frac{1}{xyz}yz\right), & \left(x\frac{1}{xyz}\cdot xz\right), & x\cdot\frac{1}{xyz}(xy) \\ \frac{-3^y}{\sqrt{1-x^2 3^{2y}}}, & \frac{-x(\ln 3)(3^y)}{\sqrt{1-x^2 3^{2y}}}, & 0 \end{pmatrix}$$

2×3 matrix.

We can multiply matrices by vectors

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

$3\times 2 \quad 2\times 1$

If M is an $n \times k$ matrix

$F(x) = Mx$ on k-dim vectors x

$F: \mathbb{R}^k \to \mathbb{R}^n$.

Use of the derivative matrix: $F' = \begin{pmatrix} (F_1)_x & (F_1)_y & \cdots \\ (F_2)_x & (F_2)_y & \cdots \\ \vdots & & \end{pmatrix}$

every entry tells us something about how the function is changing.

$\boxed{\text{Example}}$ Let $G(x,y,z) = \left(\overset{G_1}{\text{temp at } (x,y,z)}, \overset{G_2}{\text{pressure at } (x,y,z)} \right)$

(x,y,z) is a point in our room.

Suppose at a certain pt,

$$G' = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

positive: temp increasing in x direction (rate might be $2°F/ft$.)

negative: pressure decreasing as z increases.

temp is $\approx$ constant as you move in $+z$ direction.

Recall from Calc 2

Taylor series:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

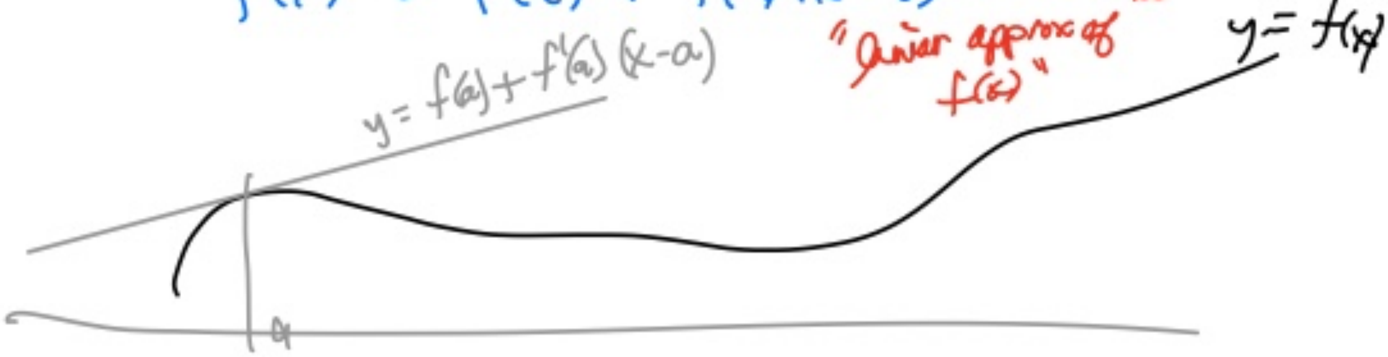
eg $f(x) = e^x$, $a = 0$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

First two terms: degree 1 Taylor polynomial

$$f(x) \approx f(a) + f'(a)(x-a)$$ ← eqn of tangent line.

"linear approx of f(x)"



Note: $f(x)$ is differentiable at a

$\iff$ $f(x)$ is very close to its tangent line at $x = a$.

Multivariable Calculus.

$$F: \mathbb{R}^k \to \mathbb{R}^n$$

$$F(x) \approx F(a) + \underbrace{F'(a)}_{n \times k \text{ matrix}}(x-a)$$ ← best linear approximation to F.

$a = (a_1, a_2, \dots, a_k)$

$$x - a = \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_k - a_k \end{pmatrix}$$

One case: $F: \mathbb{R}^k \to \mathbb{R}^1$

$$F(x_1, \ldots, x_k) = F(a) + \underset{1\times k}{F'(a)}\,\underset{k\times 1}{(x-a)}$$

$$= \underbrace{F(a) + \nabla F(a) \cdot (x-a)}_{\text{equation of the tangent hyperplane.}}$$

Note $\nabla F(a)$ is normal ($\perp$) to the tangent plane at $x = a$.

We say a function $f: \mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $x = a = (a_1, \ldots, a_k)$

matrix mult.

if
$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{|h|} = 0$$

$h = (h_1, h_2, \ldots)$

Back to Calc I calculations.

$\boxed{\text{Example}}$ Consider $y^2 x = 3y - x$.

Find the tangent line equation to this curve at $\left(\frac{3}{2}, 1\right)$.

Solution: Consider $f(x,y) = y^2 x - 3y + x = 0$

$$\nabla f = \left(y^2 + 1, \; 2xy - 3\right)$$

$= \left(2, 2\left(\tfrac{3}{2}\right)(1) - 3\right) = (2, 0).$

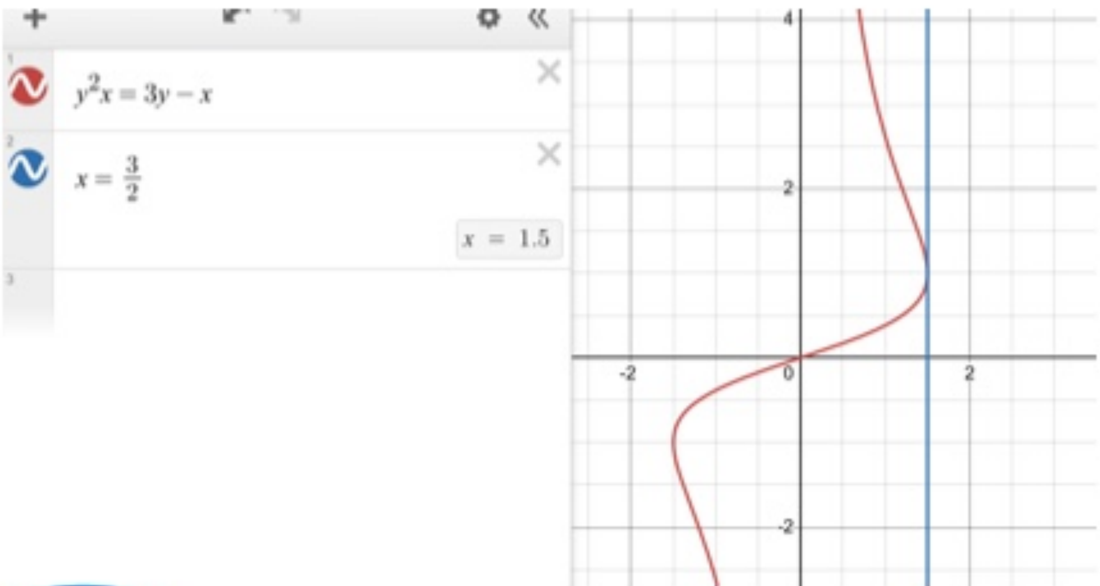
Ans

∇f is perp. to the level set $f(x,y) = 0$.

$$(2,0)\cdot(x - x_0, y - y_0) = 0$$

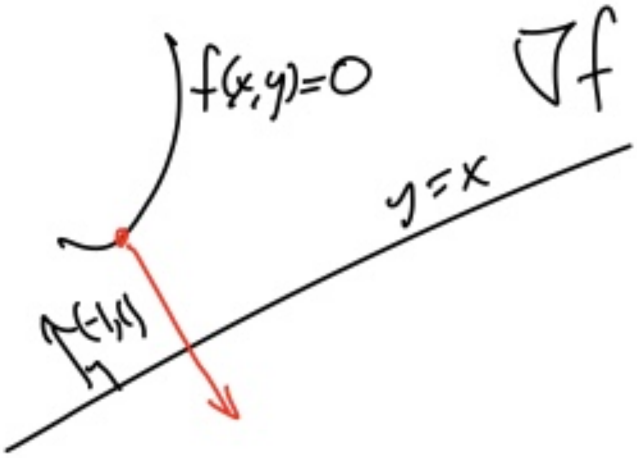
$$2\left(x - \tfrac{3}{2}\right) + 0 =$$

$$x = \tfrac{3}{2} \Rightarrow \boxed{x = \tfrac{3}{2}}$$



Example Find the points of $y^2x = 3y - x$ where the tangent line is parallel to $y = x$.

Solution: $f(x,y) = y^2x - 3y + x$



$$\nabla f = (y^2+1,\ 2xy-3)$$

$$y = x$$

$$g(x,y) = y - x = 0$$

$$\nabla g = (-1, 1)$$

Want ∇f, ∇g to be parallel

$$\iff \nabla f = c\,\nabla g$$

$$y^2+1 = c(-1) \Rightarrow$$

$$2xy - 3 = c(1) = c$$

$$2xy = 3 + c$$

$$x = \frac{3+c}{2y} \qquad y^2 = -c - 1$$

$$c = -y^2 - 1$$

$$x = \frac{3 - y^2 - 1}{2y} = \frac{2-y^2}{2y}.$$

$$y^2x - 3y + x = 0 \Rightarrow y^2\left(\frac{2-y^2}{2y}\right) - 3y + \frac{2-y^2}{2y} = 0$$

mult by $2y \Rightarrow y^2(2-y^2)-6y^2+2-y^2=0$

$$2y^2-y^4-6y^2+2-y^2=0$$

$$-y^4-5y^2+2=0$$

$$y^4+5y^2-2=0$$

$$y^2=\frac{5\pm\sqrt{25+8}}{2}$$

$$y=\pm\sqrt{\frac{5\pm\sqrt{33}}{2}}$$

must use +

$$x=\frac{2-y^2}{2y}.$$

I would get the two points!

3-d example.

Find the equation of the tangent plane to $z=x^2+4x-y^2+2y$ at the point $(1,1,6)$.

$$F(x,y,z)=x^2+4x-y^2+2y-z=0$$

$$\nabla F = (2x+4, -2y+2, -1)$$

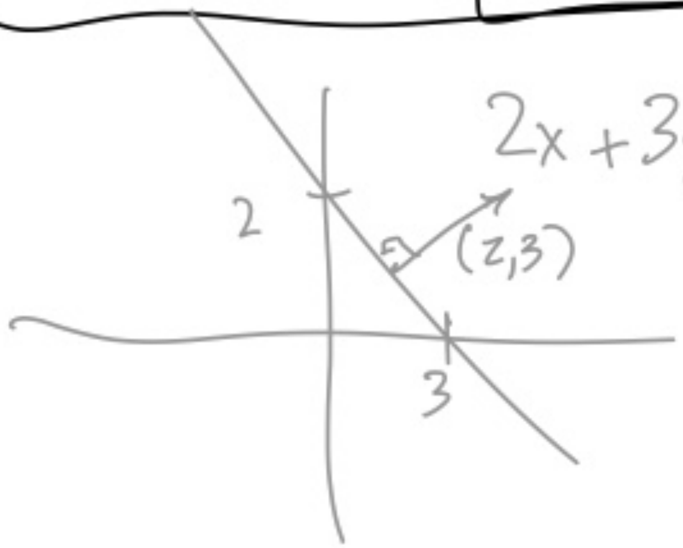
$$= (6, 0, -1)$$

$$6(x-1) + 0(y-1) + (-1)(z-6) = 0$$

$$6x - 6 - z + 6 = 0$$

$$\boxed{6x - z = 0}$$

$2x + 3y = 6$



$(2,3)$ is normal to this line.